Compensating amplitude-dependent tune-shift without driving fourth order resonances

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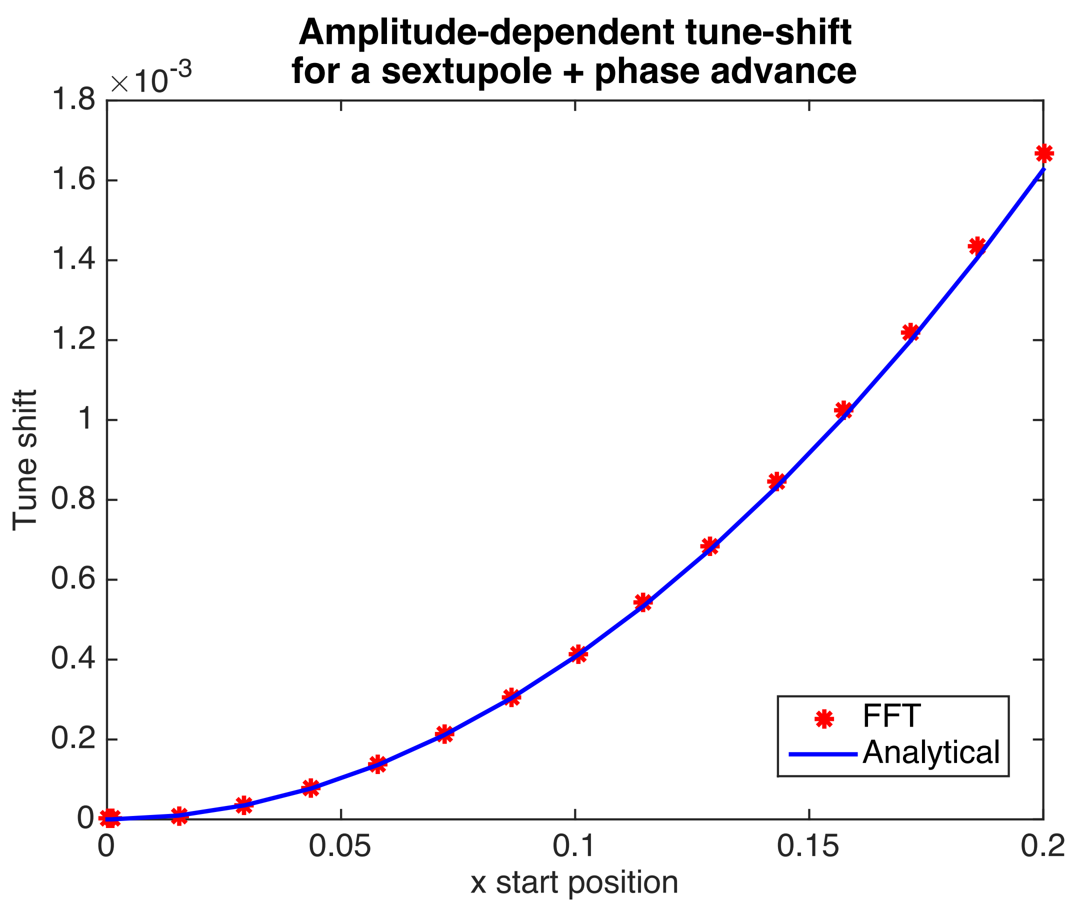
Introduction

Sextupoles are often introduced to a ring to control chromaticity. But they also drive amplitude-dependent tune-shift in second order.

This tune-shift is proportional to the action, i.e. proportional to monomials:

$$(x^2 + x'^2)^2 \qquad (y^2 + y'^2)^2 \qquad (x^2 + x'^2)(y^2 + y'^2) \qquad\qquad J_x^2,\ \ J_y^2,\ \ J_x J_y$$



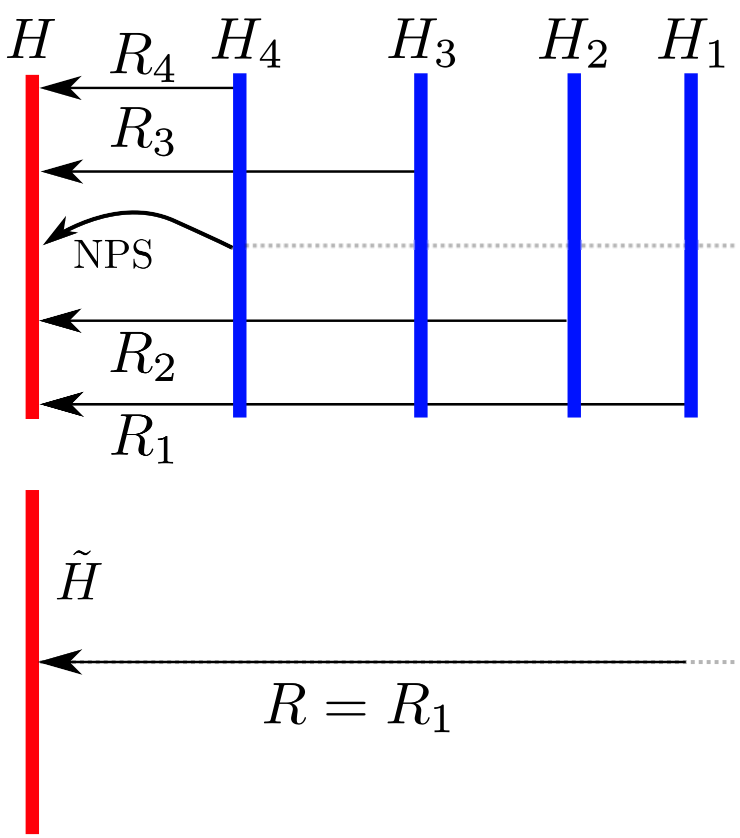
- Sextupoles drive amplitude-dependent tune-shift
- We can use octupoles to compensate

$$H_{\text{oct}} = \frac{k_3}{4!}(x^4 - 6x^2y^2 + y^4)$$

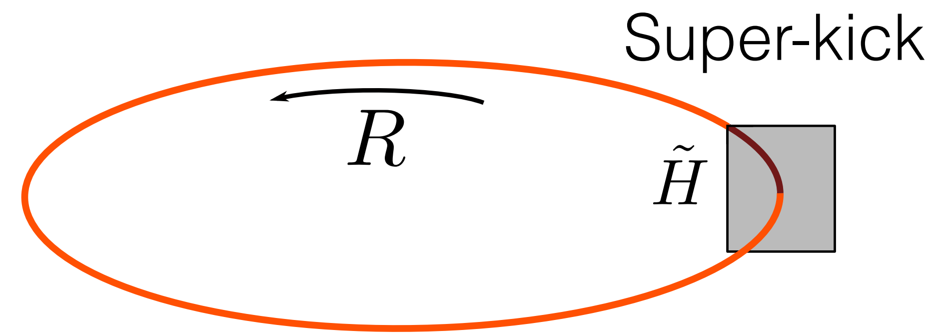
- But octupoles drive additional resonances

Method

We can move Hamiltonians using a **similarity transformation** and then concatenate to an effective Hamiltonian using the **Campbell-Baker-Hausdorff** formula.



First move H_4 and concatenate with H, then move H_3 etc.



A full turn map in normal form representation:

$$\mathcal{M} = \mathrm{e}^{:-\tilde{H}:}R = \mathrm{e}^{:-K:}\mathrm{e}^{:-C:}R\mathrm{e}^{:K:}$$

Adding octupoles only contribute linearly to fourth order:

$$C^{(4)} = \mathrm{Pr}\left\{H_{\mathrm{oct}}^{(4)} + \tilde{H}^{(4)} + \frac{1}{2}\left[\tilde{H}^{(3)}, S^{(3)}K^{(3)}\right]\right\}$$

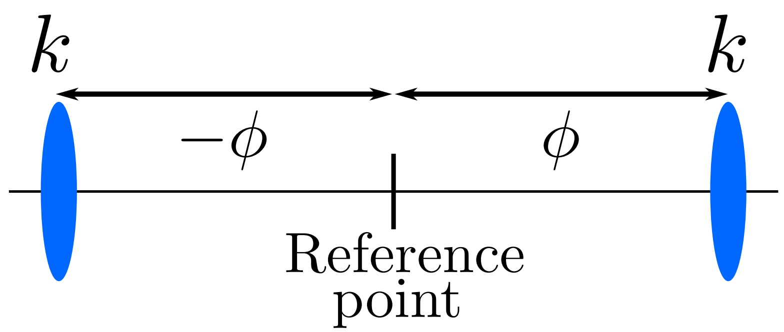
To compensate tune-shift: set octuple strengths such RHS = 0.

We have a MATLAB-code for polynomial representation that can do CBH, Normal forms etc. and we compare the results with tracking.

Optimum placement of octuples

We start with two octuples (horizontal motion only) and write the Hamiltonians in action-angle variables and move both Hamiltonians to the reference point via the similarity transformation:

$$\begin{aligned}
\tilde{H} &= k(x\cos\phi + x'\sin\phi)^4 + k(x\cos\phi - x'\sin\phi)^4 \\
&= k\left[x^4\cos^4\phi + 4x^3x'\cos^3\phi\sin\phi + 6x^2x'^2\cos^2\phi\sin^2\phi\right. \\
&\quad \left.+4xx'^3\cos\phi\sin^3\phi + x'^4\sin^4\phi\right] \\
&+ k\left[x^4\cos^4\phi - 4x^3x'\cos^3\phi\sin\phi + 6x^2x'^2\cos^2\phi\sin^2\phi\right. \\
&\quad \left.-4xx'^3\cos\phi\sin^3\phi + x'^4\sin^4\phi\right] \\
&= 2k\left\{x^4\cos^4\phi + 6x^2x'^2\cos^2\phi\sin\phi + x'^4\sin^4\phi\right\}
\end{aligned}$$



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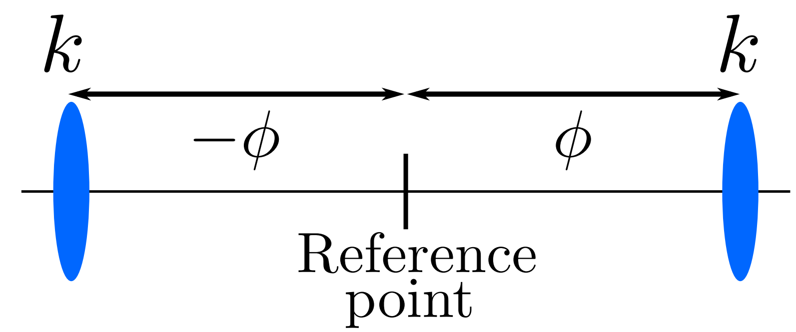
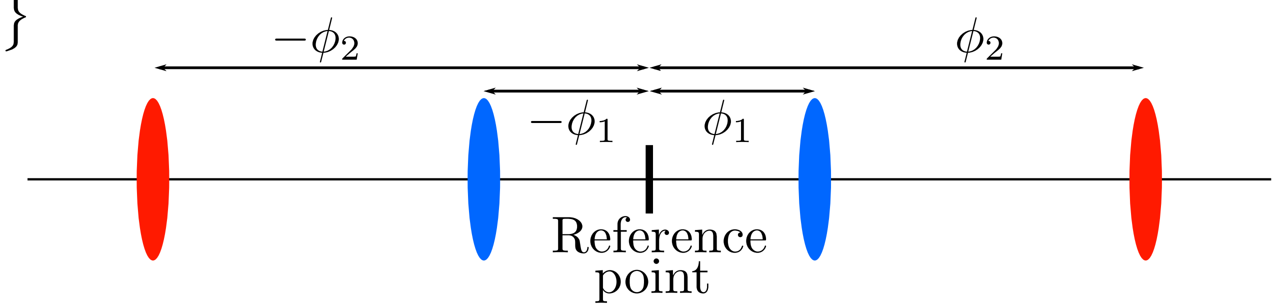
&+ k\left[x^4\cos^4\phi - 4x^3x'\cos^3\phi\sin\phi + 6x^2x'^2\cos^2\phi\sin^2\phi \right. \\
&\qquad \left. -4xx'^3\cos\phi\sin^3\phi + x'^4\sin^4\phi\right] \\
&= 2k\left\{x^4\cos^4\phi + 6x^2x'^2\cos^2\phi\sin\phi + x'^4\sin^4\phi\right\}
\end{aligned}
$$





Short-hand notation:

$$c_1 = \cos\phi_1 \qquad s_1 = \sin\phi_1 \qquad \text{etc.}$$

Move all four octupoles to reference point:

$$
\begin{aligned}
\bar{H} &= 2k_1\left[x^4c_1^4 + 6x^2x'^2c_1^2s_1^2 + x'^4s_1^4\right] + 2k_2\left[x^4c_2^4 + 6x^2x'^2c_2^2s_2^2 + x'^4s_2^4\right] \\
&= 2x^4(k_1c_1^4 + k_2c_2^4) + 12x^2x'^2(k_1c_1^2s_1^2 + k_2c_2^2s_2^2) + 2x'^4(k_1s_1^4 + k_2s_2^4)
\end{aligned}
$$

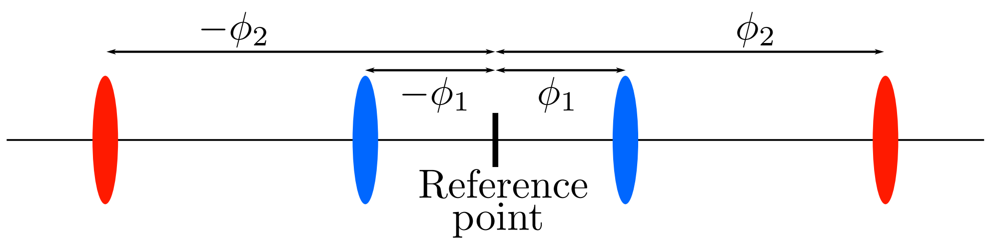
On the form $x^4 + x^2x'^2 + x'^4$. Terms with x^3x' and xx'^3 etc. cancel due to symmetry => do not drive resonances.

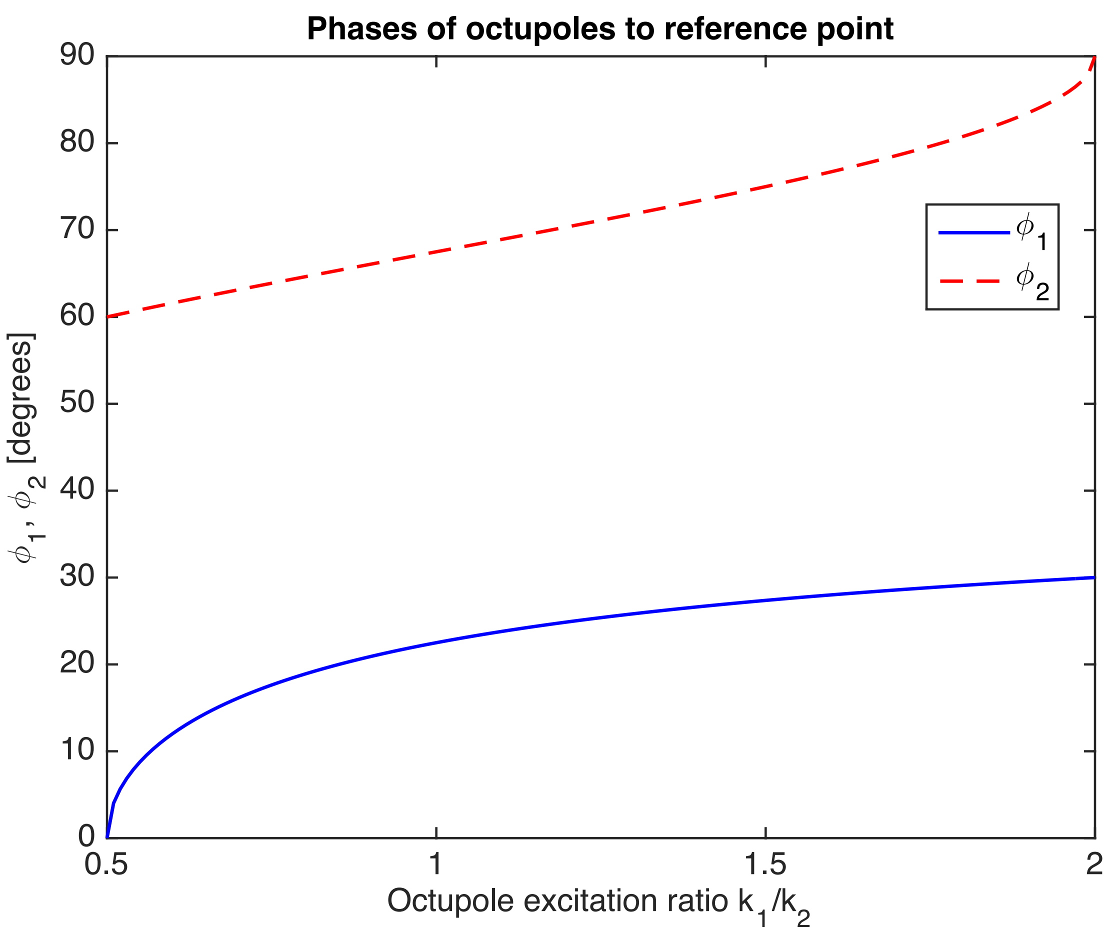
Optimum placement of octuples cont'd

In order to compensate the amplitude-dependent tune-shift we need terms containing:

$$(x^2 + x'^2)^2 = x^4 + 2x^2x'^2 + x'^4$$

This gives us a relation between k_1/k_2 and the phase advances:





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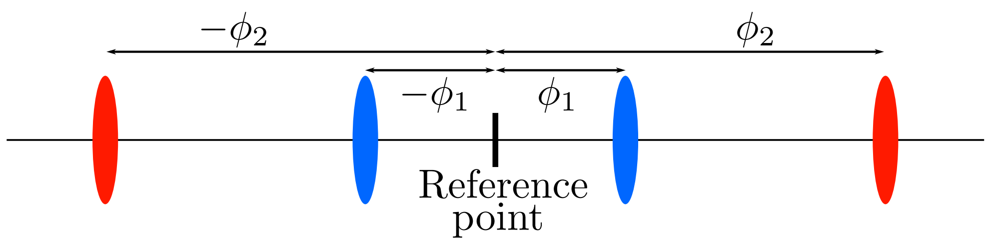
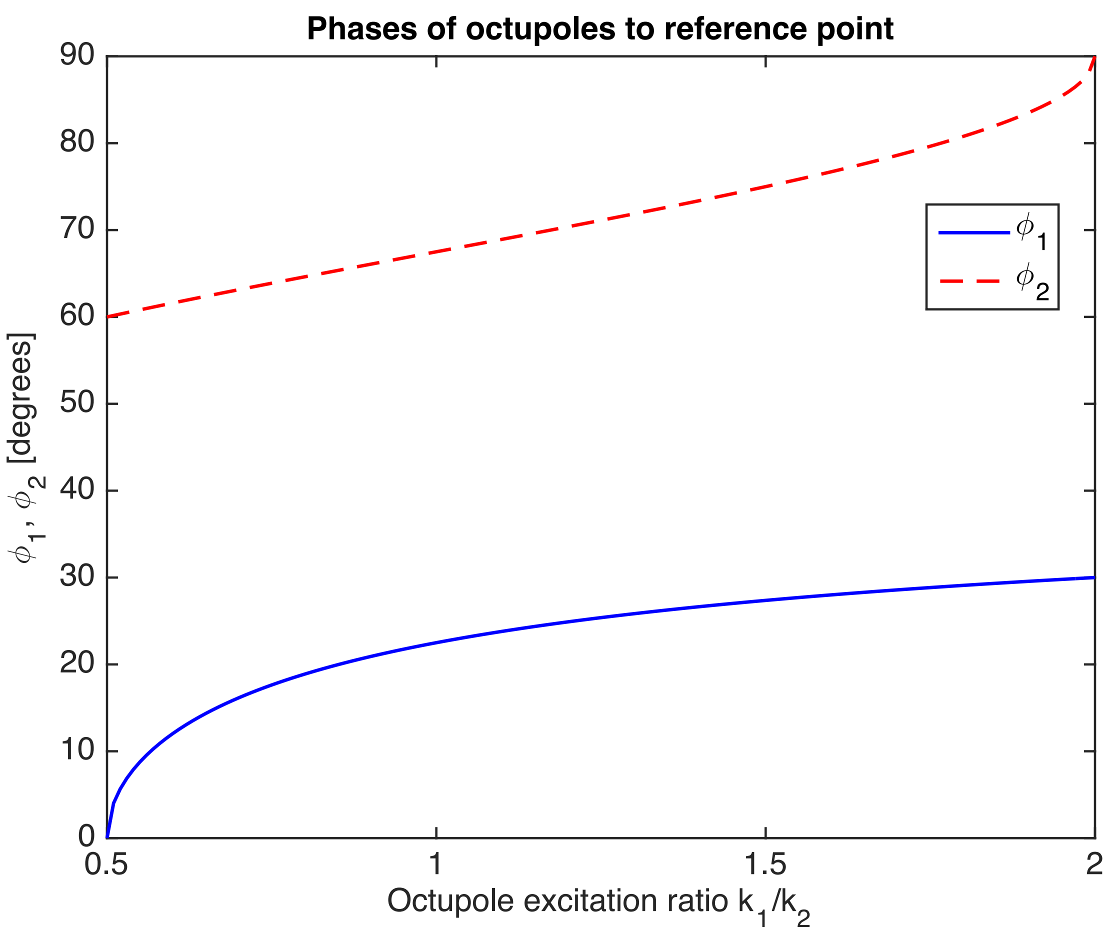
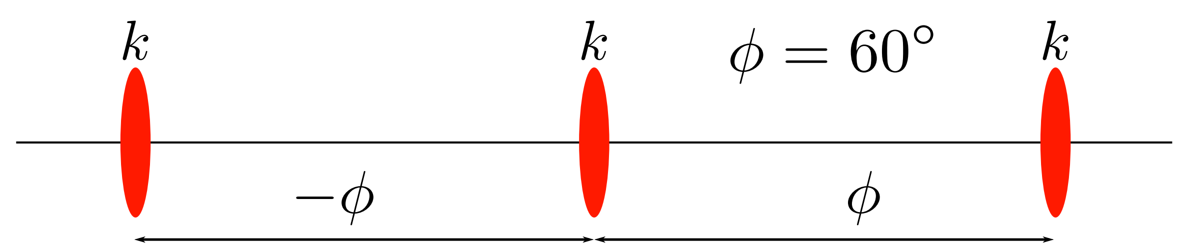




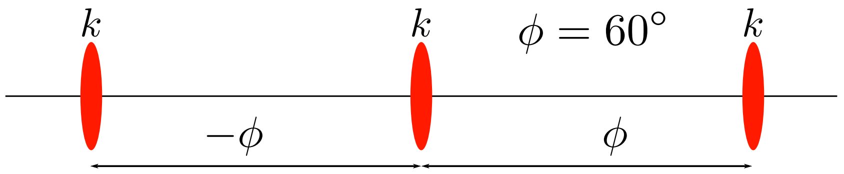
There is a solution with three equally powered octupoles and 60 degrees phase advance:



Optimum placement of octuples cont'd

The 4D Hamiltonian for an octupole in real phase space: $x = \sqrt{\beta_x}\tilde{x}$ $\quad y = \sqrt{\beta_y}\tilde{y}$

$$H = k\left(\beta_x^2\tilde{x}^4 - 6\beta_x\beta_y\tilde{x}^2\tilde{y}^2 + \beta_y^2\tilde{y}^4\right) = k_x\tilde{x}^4 - 6k_{xy}\tilde{x}^2\tilde{y}^2 + k_y\tilde{y}^4$$



Carrying out the same procedure as before (action-angle variables etc.) for the triplet we get:

$$\tilde{H} = \frac{9}{2}\left[k_x J_x^2 + k_y J_y^2 - 4k_{xy}J_xJ_y - 2k_{xy}J_xJ_y\cos(2\psi_x - 2\psi_y)\right]$$

Optimum placement of octuples cont'd

The 4D Hamiltonian for an octupole in real phase space: $x = \sqrt{\beta_x}\tilde{x}$ $\quad y = \sqrt{\beta_y}\tilde{y}$

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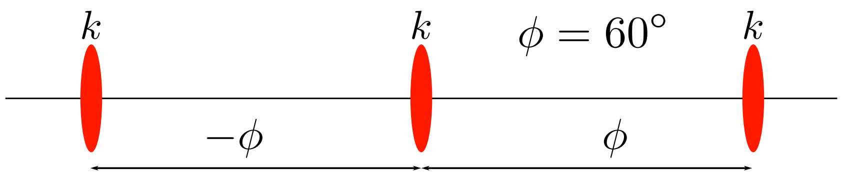
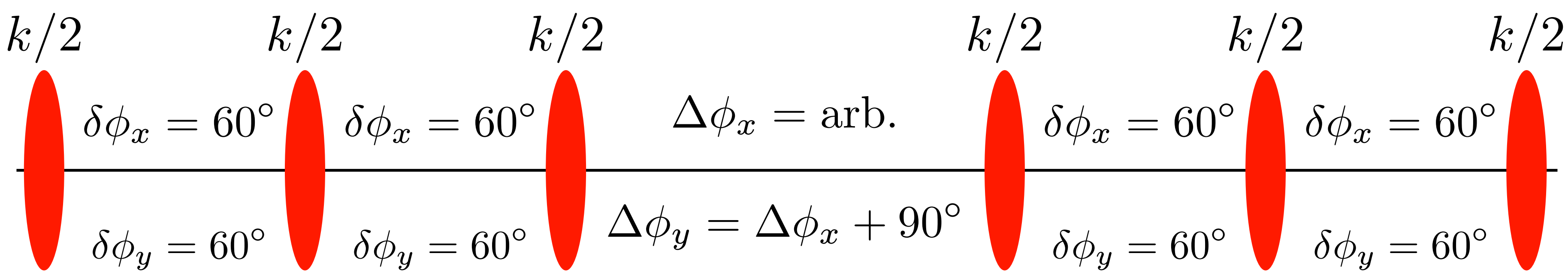


$$\tilde{H} = \frac{9}{2}\left[k_x J_x^2 + k_y J_y^2 - 4k_{xy}J_xJ_y - 2k_{xy}J_xJ_y\cos(2\psi_x - 2\psi_y)\right]$$

This drives the $2Q_x$ - $2Q_y$ resonance. In 2D we see that this setup cancel all resonances except one. We solve this by adding another triplet, i.e. a "six-pack":



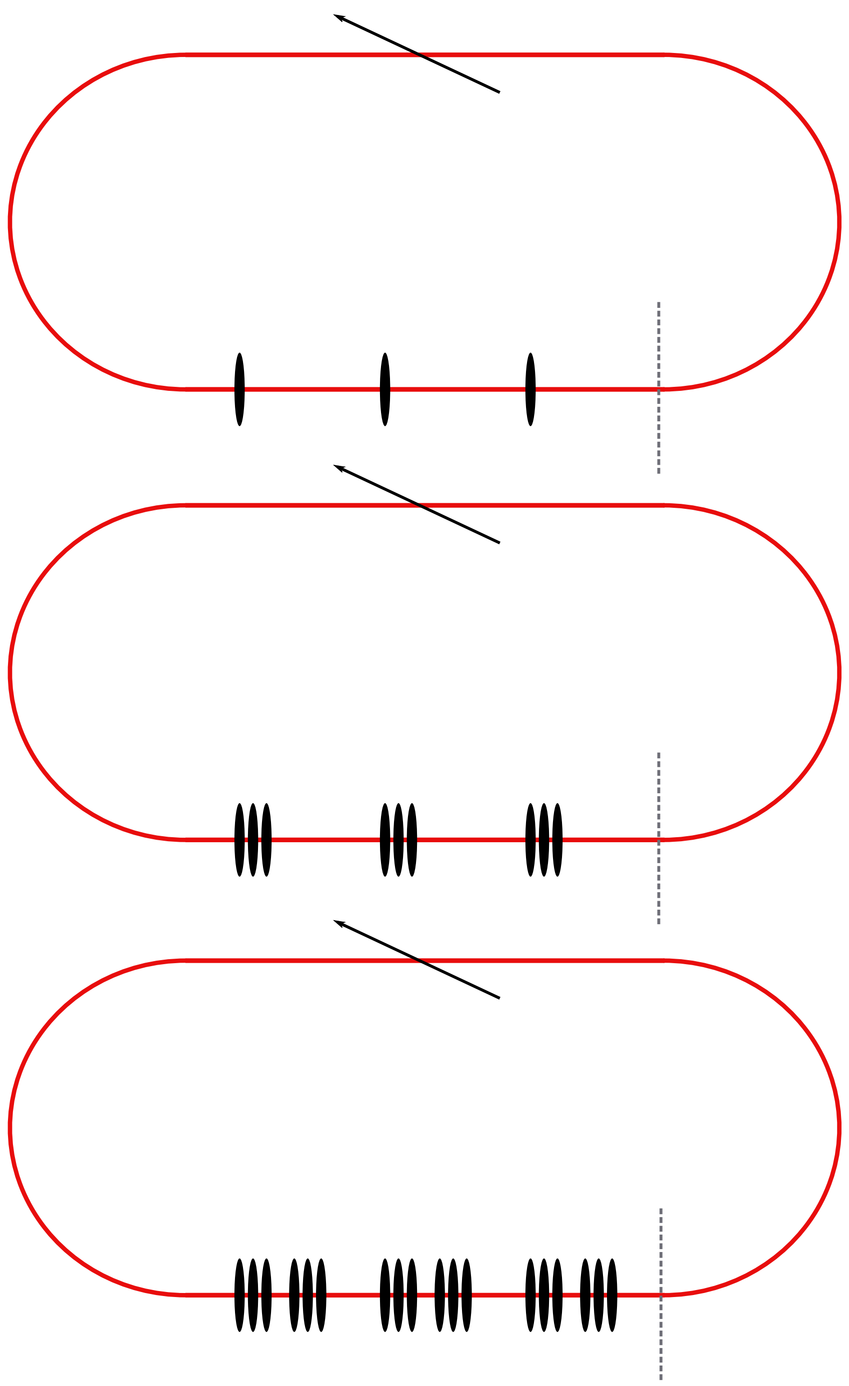
This setup can cancel a given tune-shift term without driving any fourth order resonances! In order to control all three tune-shift terms independently we need three six-packs at locations with different ratios of β_x/β_y.

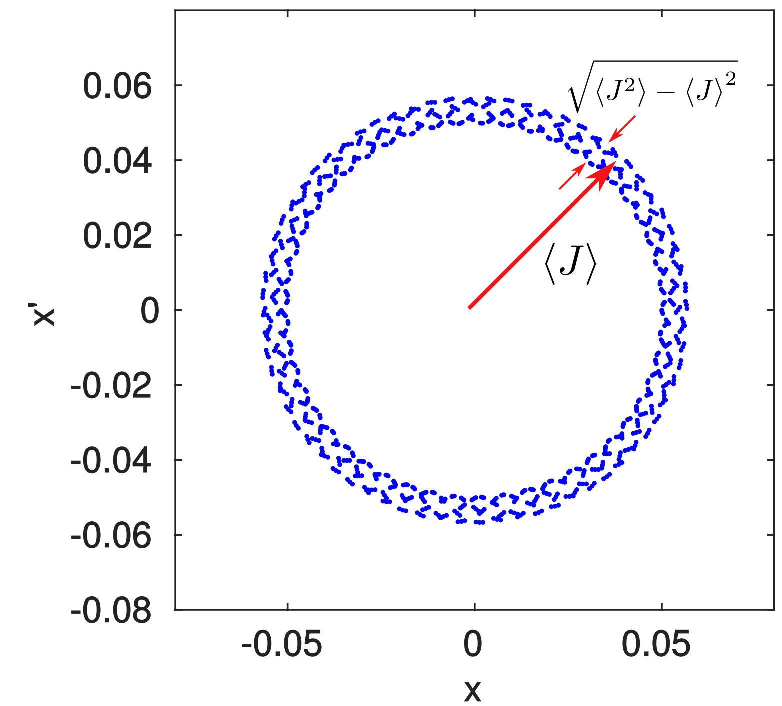
Simulation: Octupoles + phase advance

A simple setup with three setups of octupoles + a phase advance:

Smear:

$$\sigma_J = \sqrt{\frac{\langle J^2 \rangle - \langle J \rangle^2}{\langle J \rangle^2}}$$



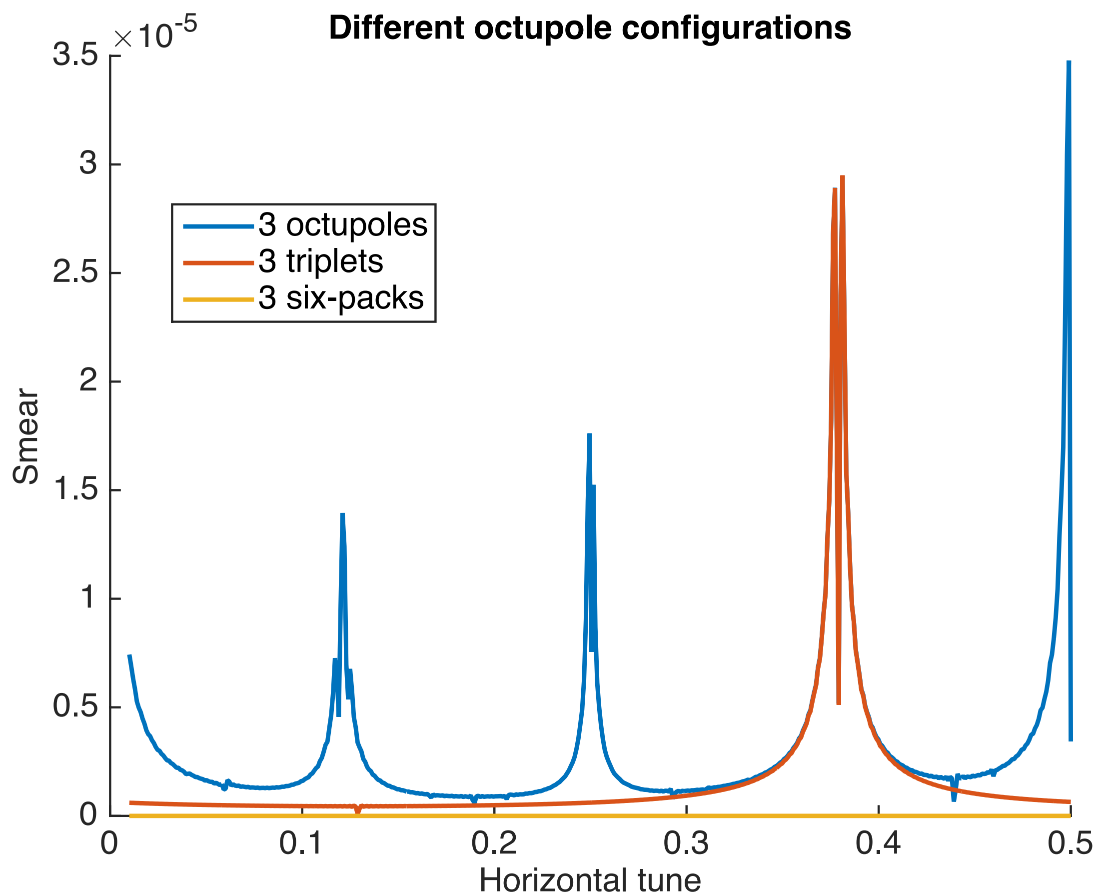


3 octupoles

3 triplets

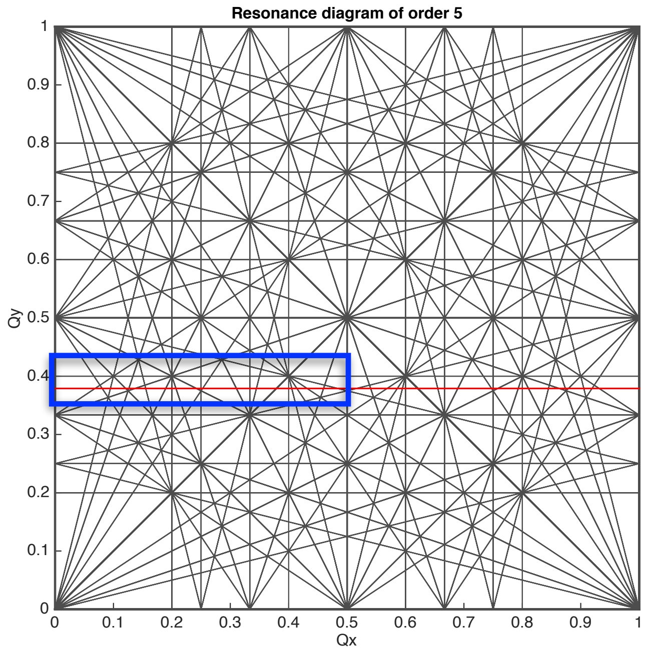
3 six-packs

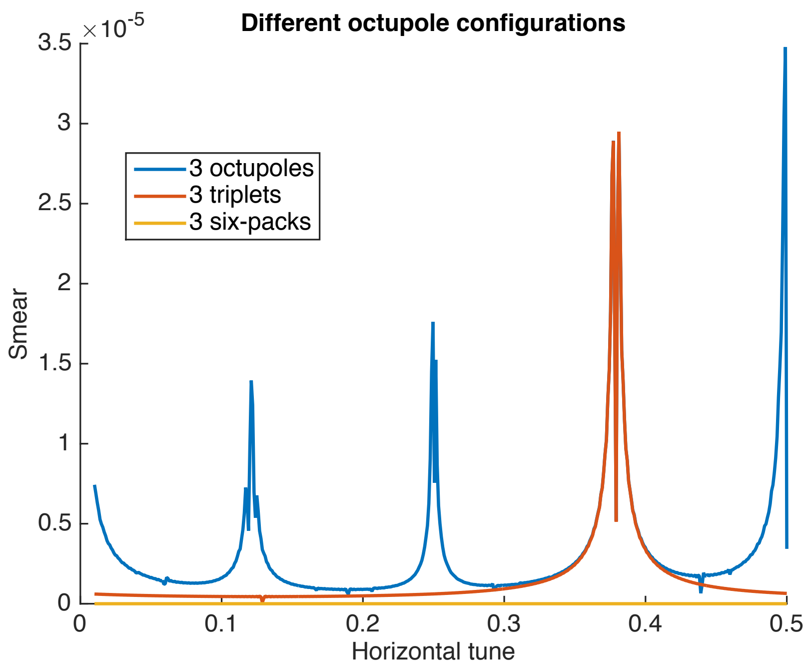
Smear plots to see resonances



Resonances

Plot smear on top of tune diagram to identify resonances





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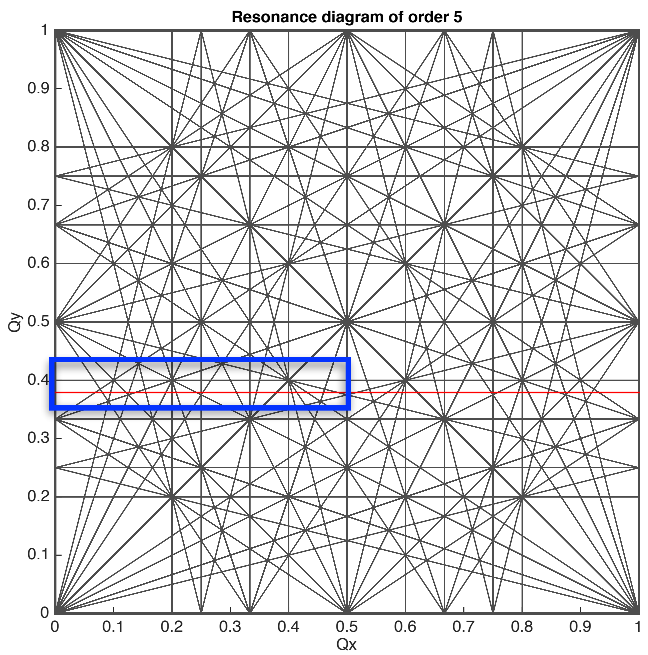

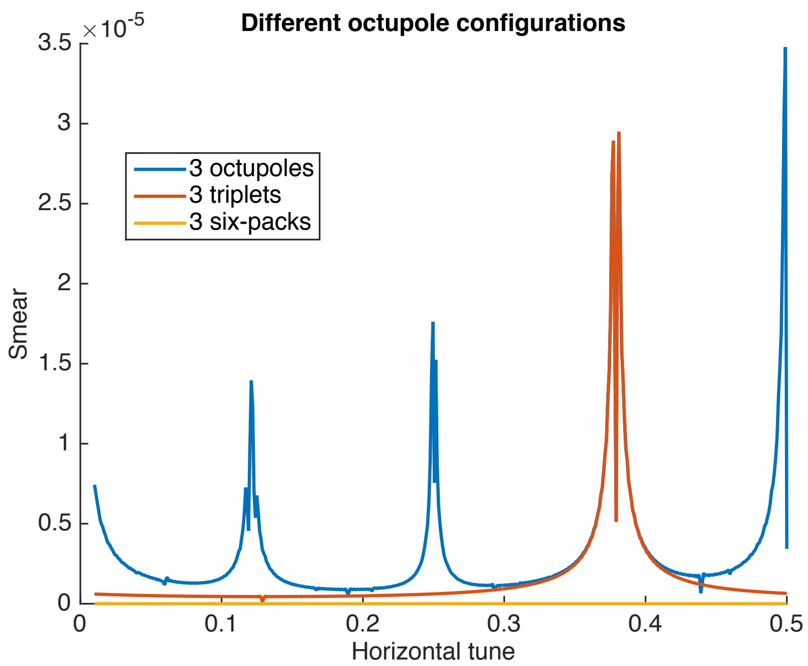

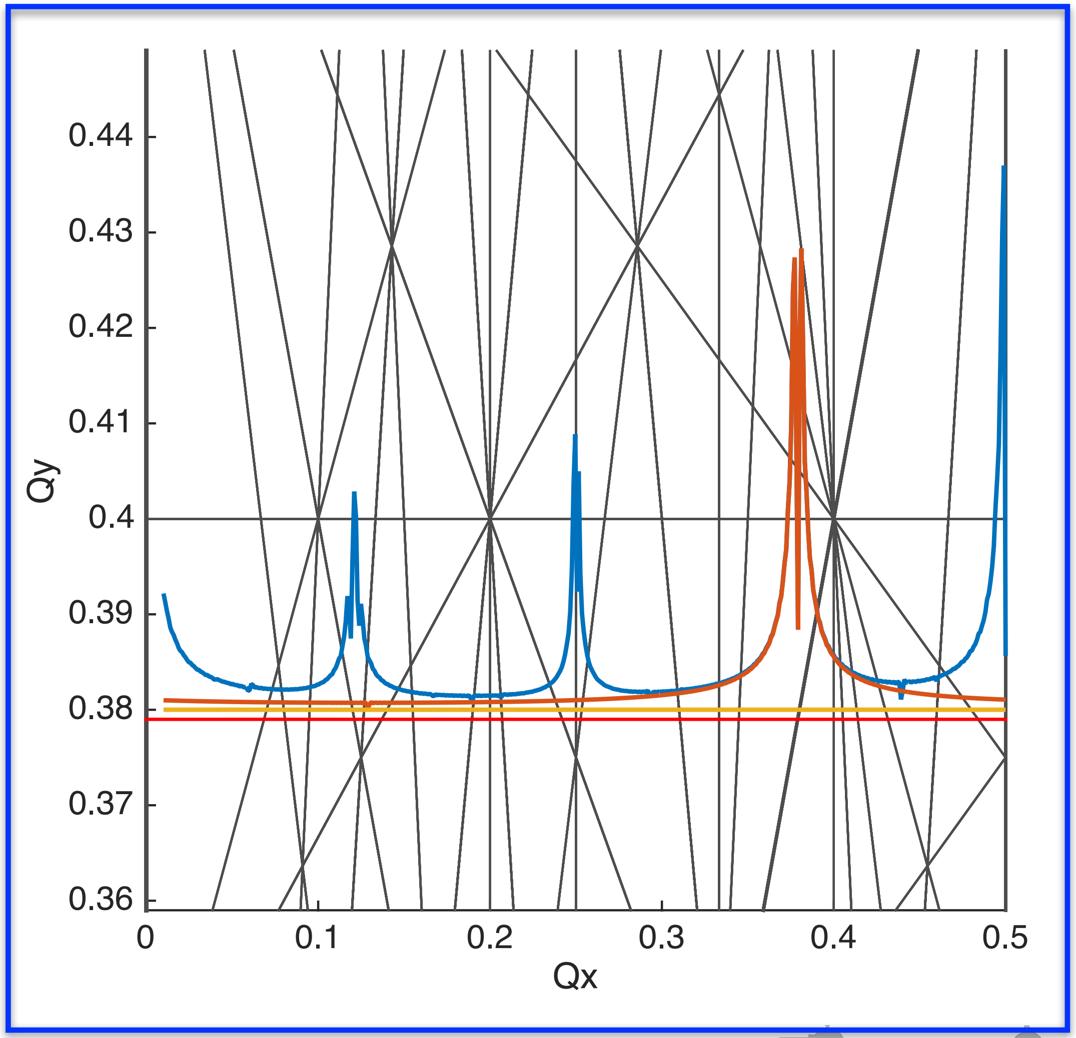

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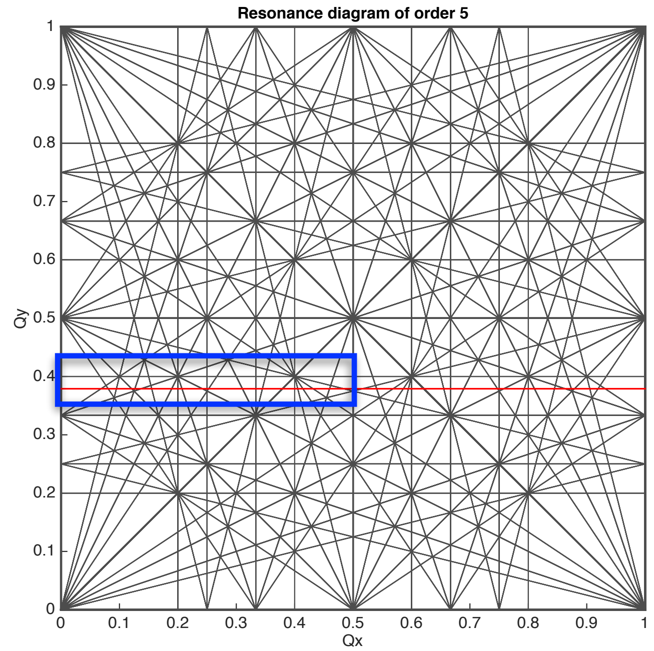
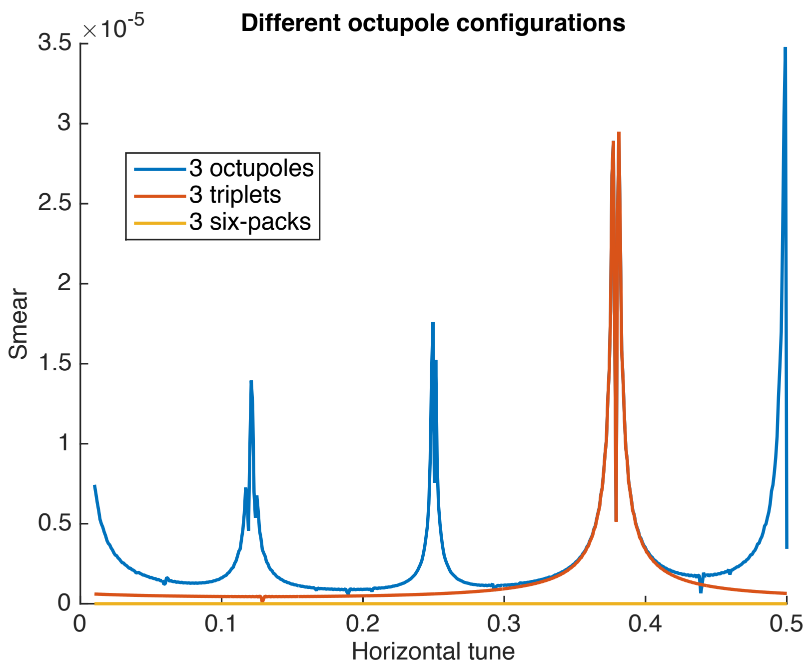
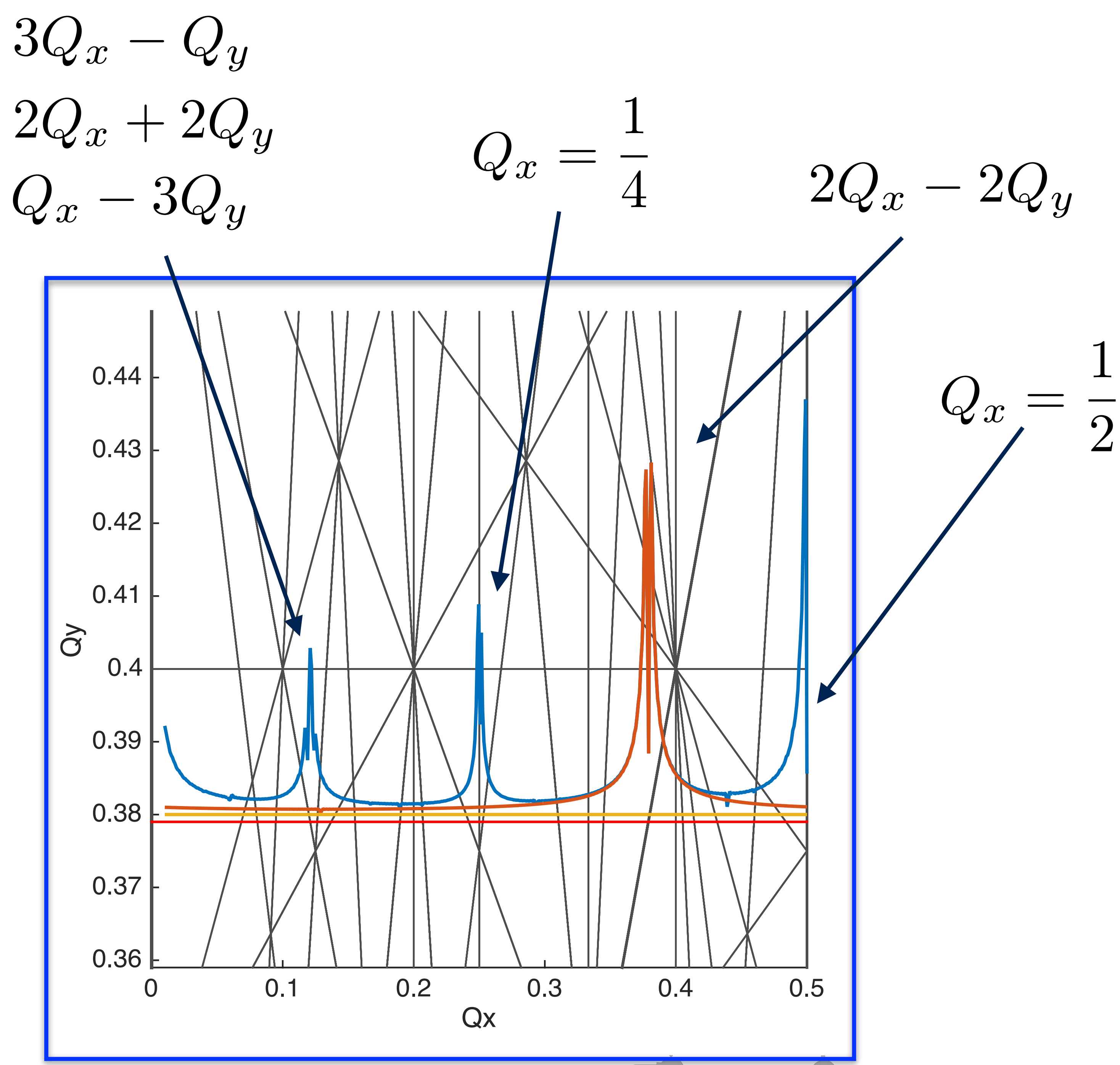
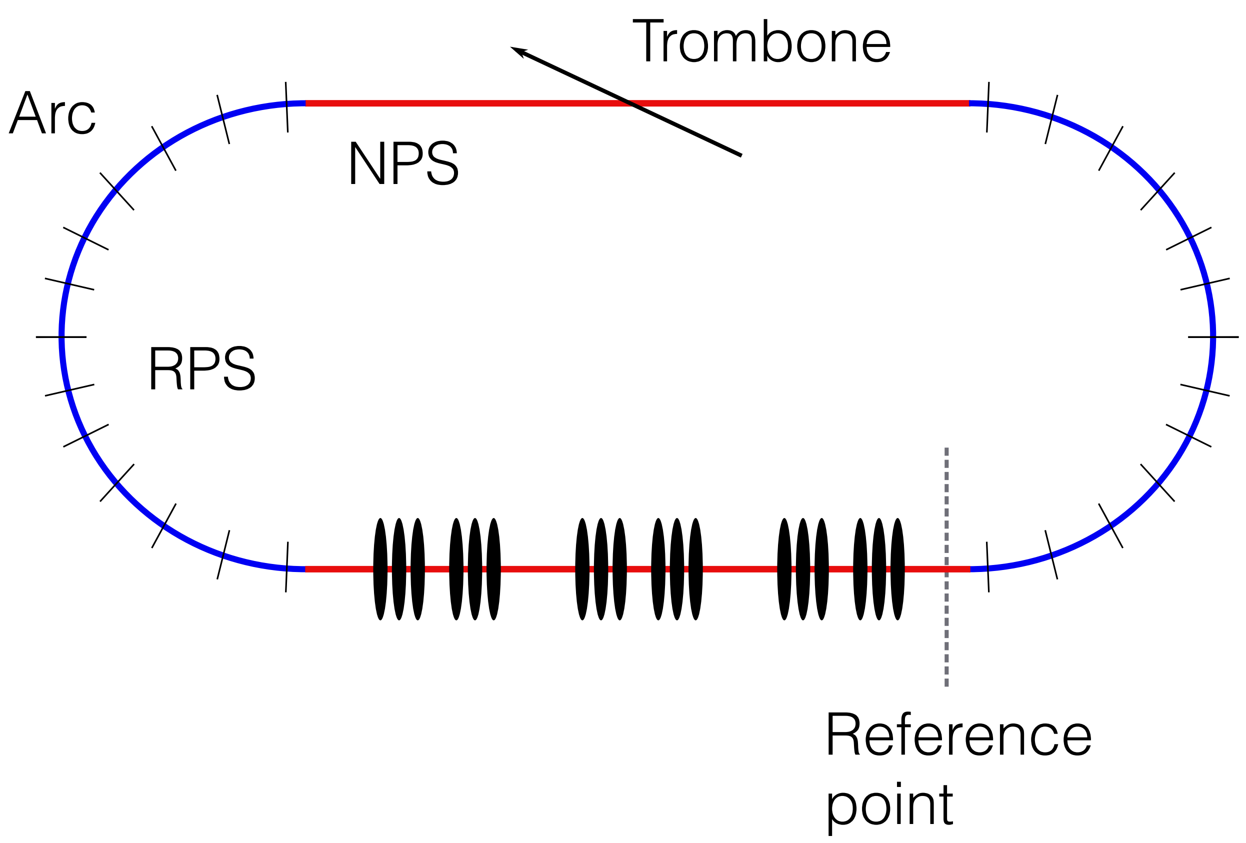






Simulation - extended model

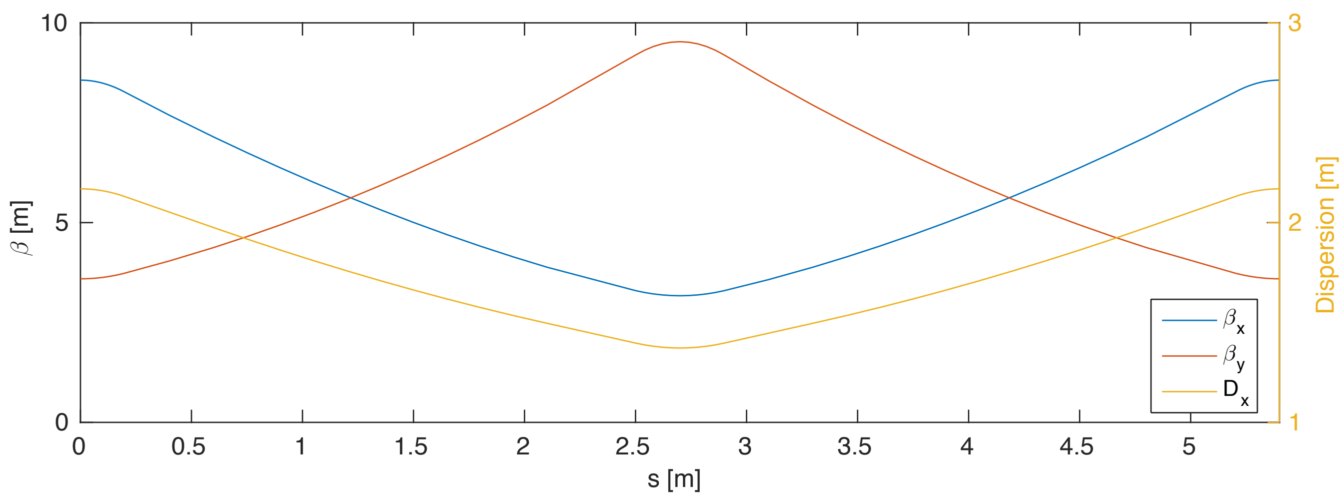


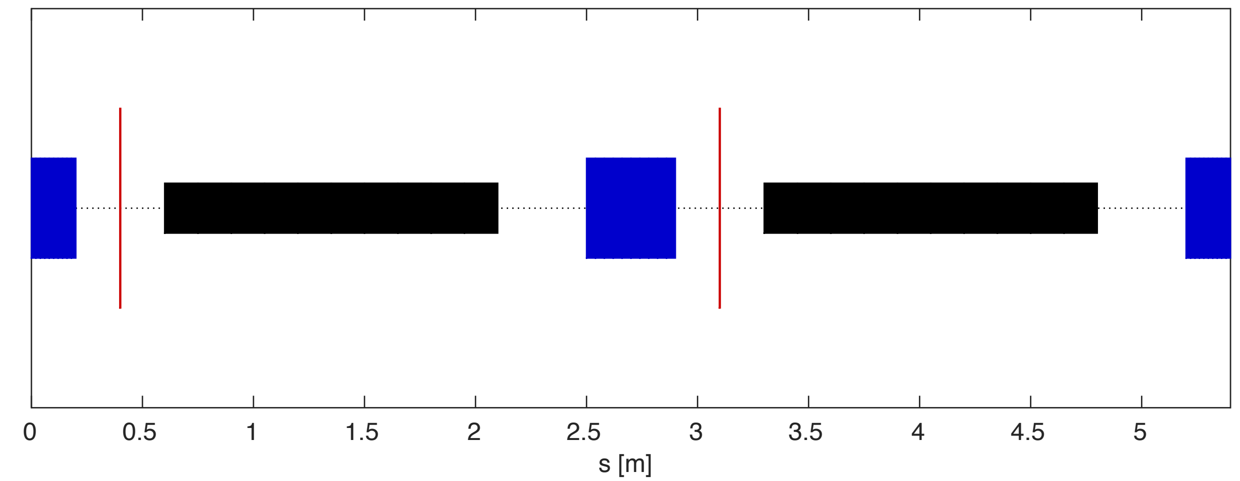
Each arc consists of 9 FODO cells.

The FODO cells include:
- 2 dipole bends
- 2 sextupoles for chromaticity correction

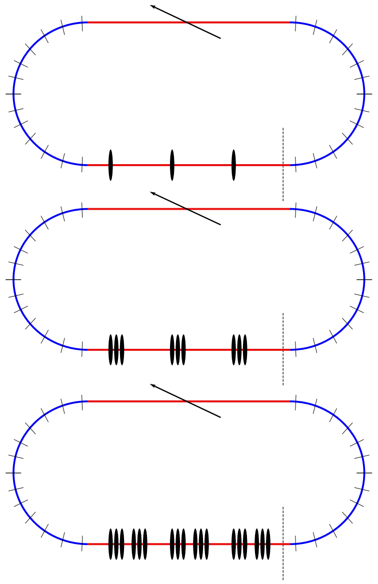
Two straight sections (NPS):

- a"trombone" for setting the overall tune
- a section containing the octupoles
- Simulate the three different octupole configurations

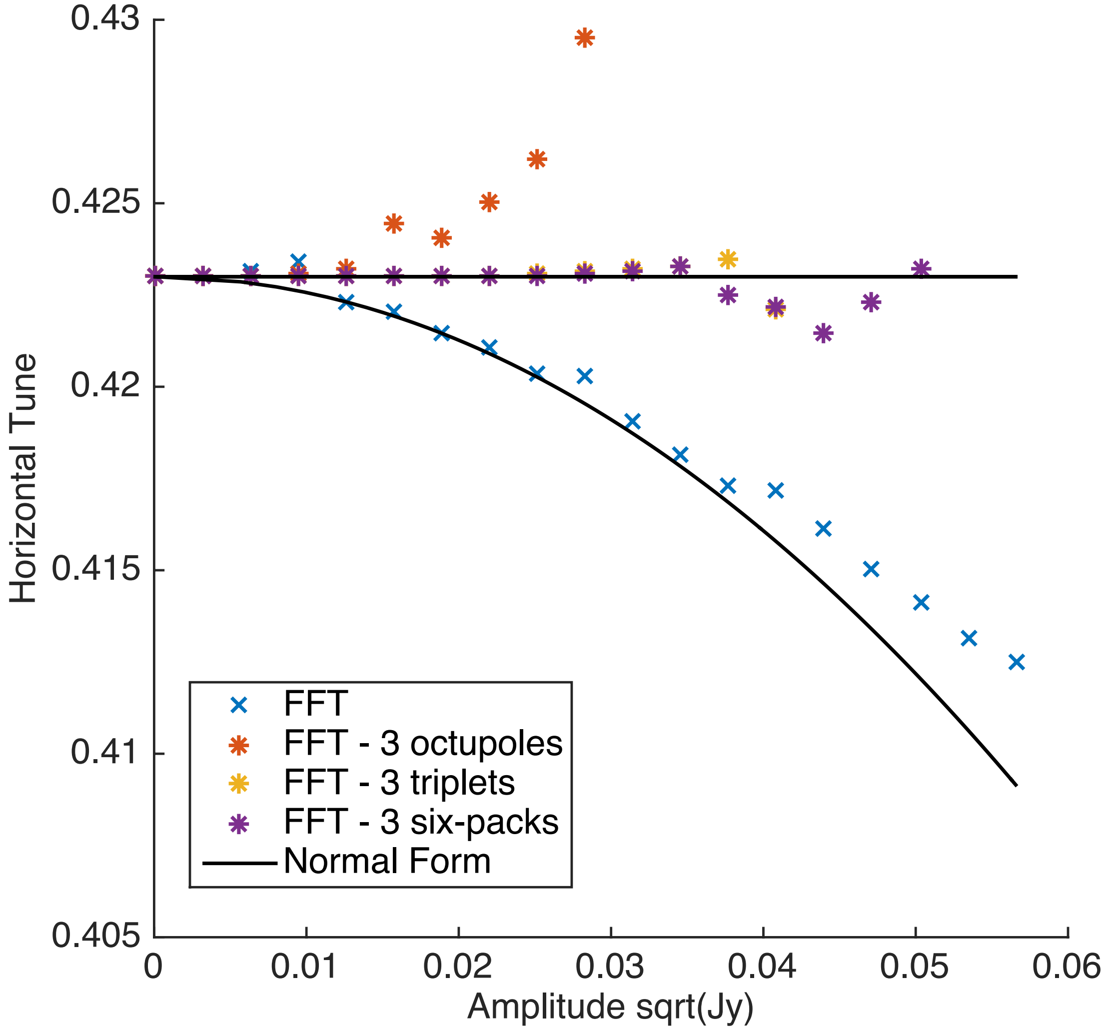


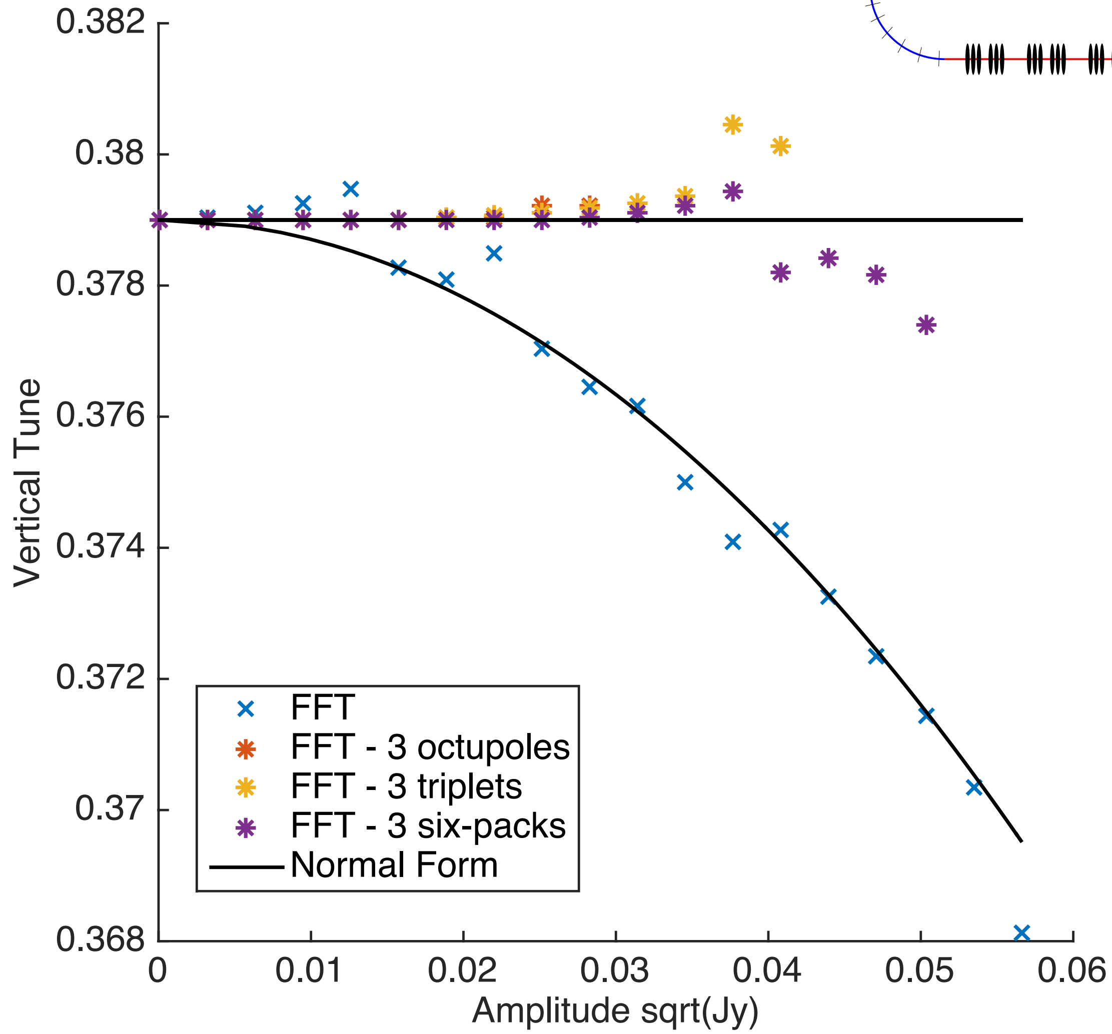


Simulation - results



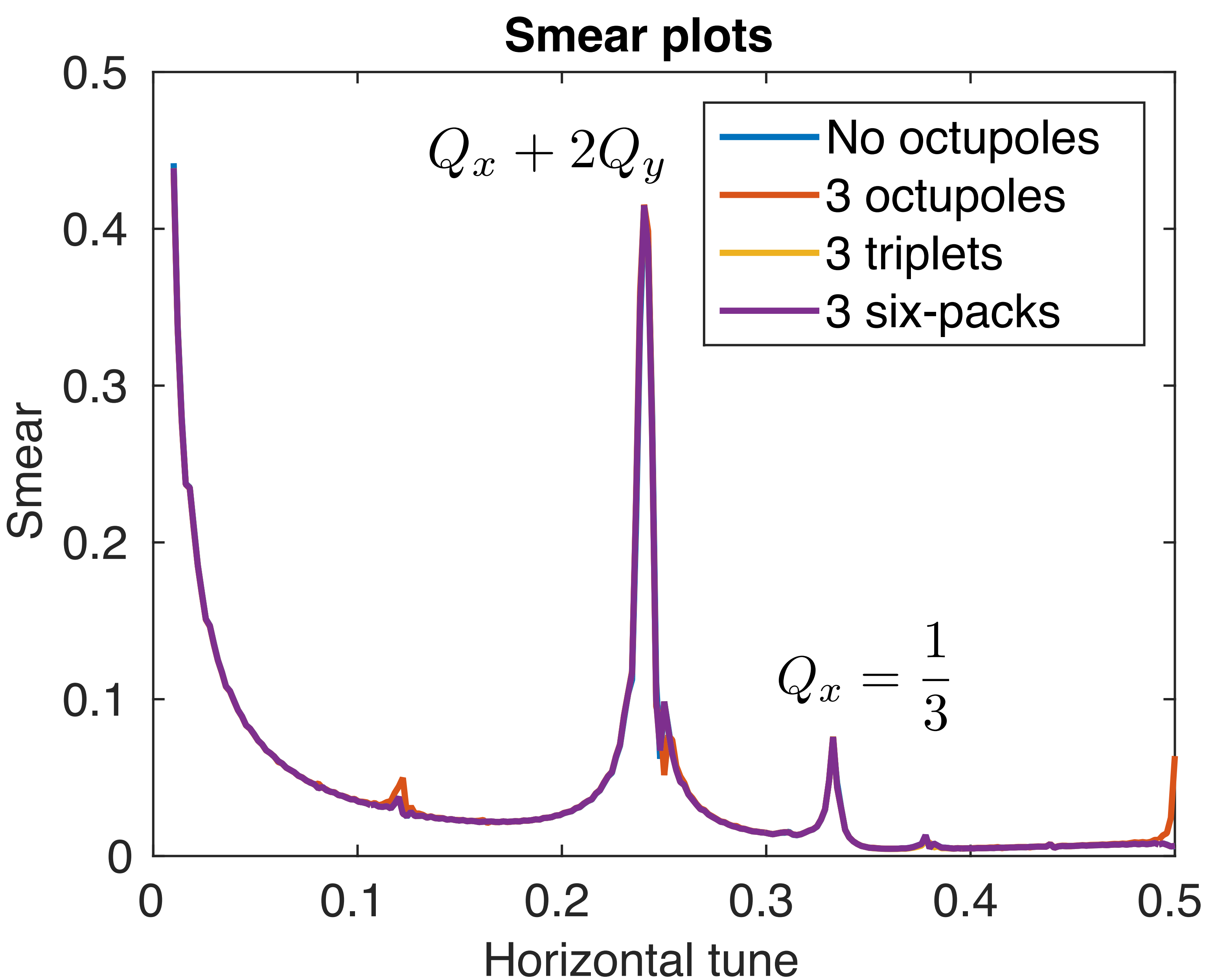
Tune-shift for the different octuple configurations:





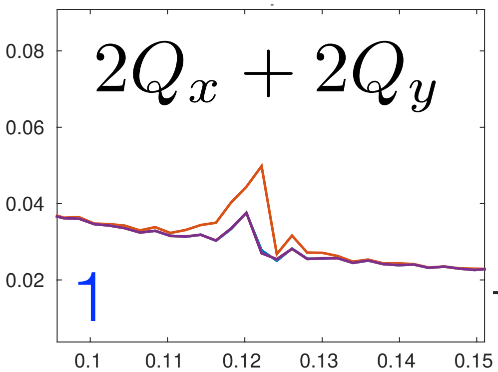
Configuration with 3 octupoles reduces stability.
Using triplets is more stable and six-packs even more so.

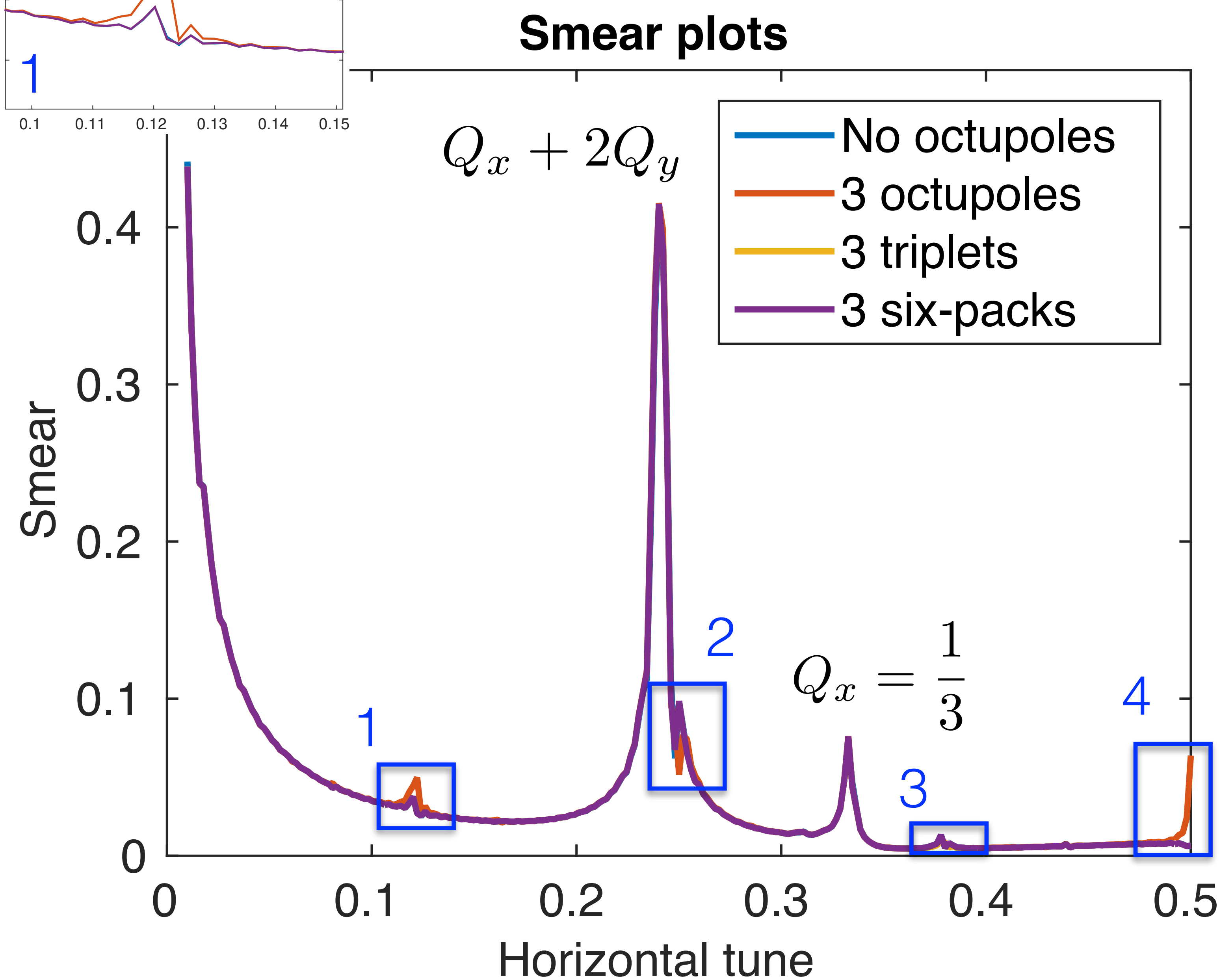
Simulation - smear



- Configuration with only three octupoles introduces some additional resonances.
- Six-packs do not add resonances.
- For this case resonances are dominated by the sextupoles.

Simulation - smear





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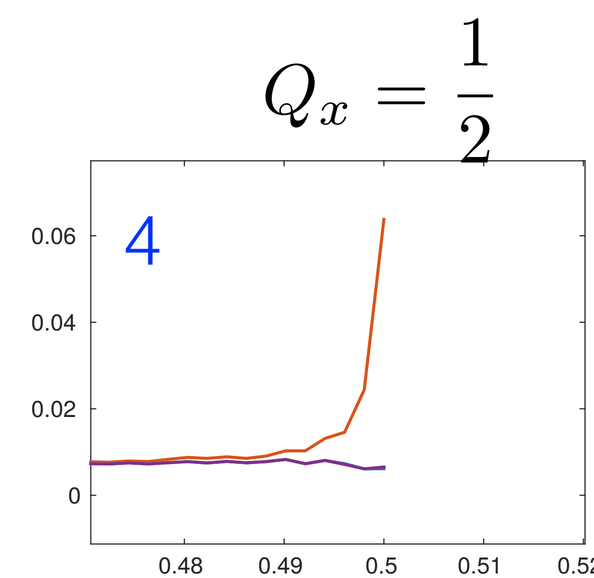
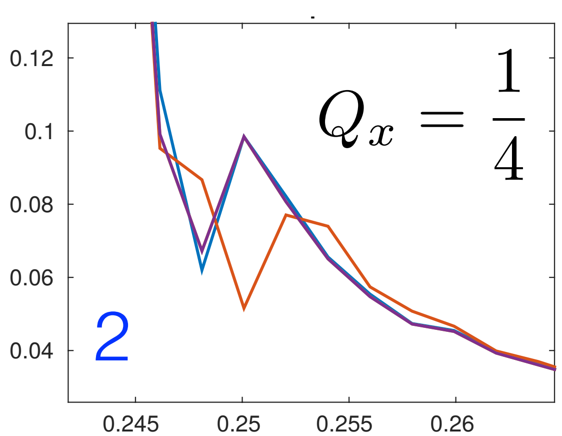
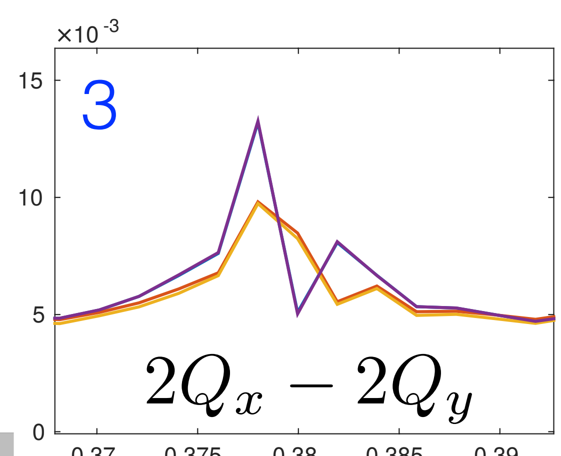






Conclusions

- Code to treat Hamiltonians and normal forms
- Powerful analytical method to understand what resonances are driven and how cancellations happen
- Used this to find optimum placement of octupoles for tune-shift compensation without driving fourth order resonances

Future work

- Include resonant normal forms to tune individual resonance-driving terms
- Knobs for compensating other resonance terms
- Apply method to an actual machine
- …

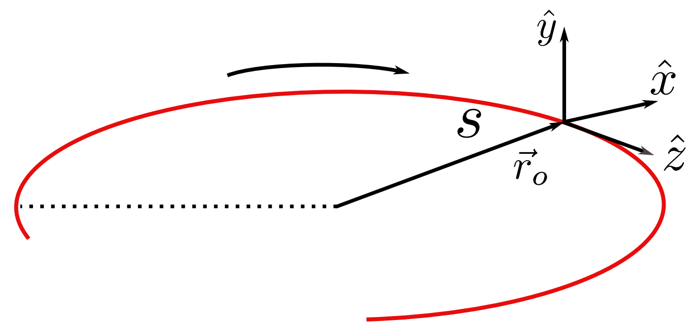
Thank you for your attention!

Backup slides

Hamiltonians

A Hamiltonian H together with Hamiltons equations describes a particle trajectory.

$$\frac{dx}{ds} = \frac{\partial H}{\partial x'} \quad ; \quad \frac{dx'}{ds} = -\frac{\partial H}{\partial x}$$



Or expressed using the Poisson bracket:

$$[f, g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial x'} - \frac{\partial f}{\partial x'}\frac{\partial g}{\partial x}$$

Then Hamilton's equations can be written as:

$$\frac{dx}{ds} = [-H, x] \quad ; \quad \frac{dx'}{ds} = [-H, x']$$

Ex: Hamiltonians for sextupole and octupole (thin elements):

$$H_{\mathrm{sext}} = \frac{k_2}{3!}(x^3 - 3xy^2)$$

Third order

$$H_{\mathrm{oct}} = \frac{k_3}{4!}(x^4 - 6x^2y^2 + y^4)$$

Fourth order

Nonlinear maps

The **Lie operator**

$$:f:g = [f,g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial x'} - \frac{\partial f}{\partial x'}\frac{\partial g}{\partial x}$$

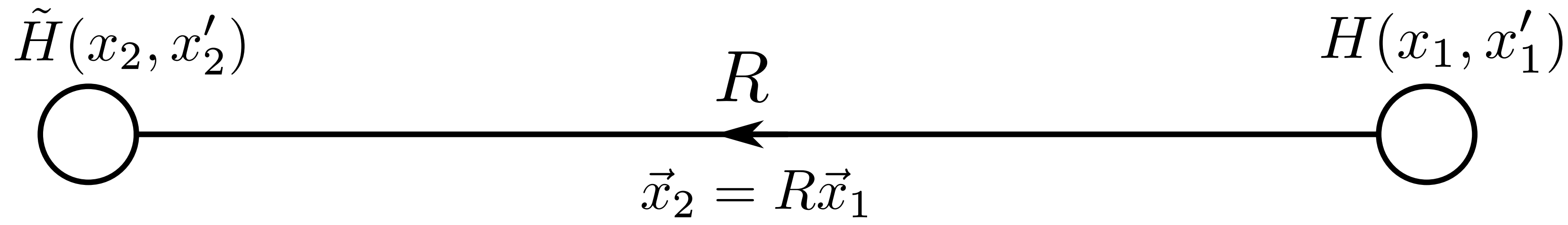
The Lie operator *f* on *g* is the Poisson bracket.

We can can calculate the change of a particle passing through an element with Hamiltonian *H* by a **Lie transformation** of the coordinate function:

$$\bar{x} = \mathrm{e}^{-:H:}x = x - [H,x] + \frac{1}{2!}[H,[H,x]] + \ldots$$

Which essentially is a Taylor map. The Lie transformation maps incoming coordinates to outgoing coordinates for a nonlinear element described by Hamiltonian *H.*

Lie Algebra



Similarity transformation:

$$\begin{aligned}\mathcal{M} &= R\mathrm{e}^{:-H(\vec{x}_1):} \\ &= R\mathrm{e}^{:-H(\vec{x}_1):}R^{-1}R \\ &= \mathrm{e}^{:-H(R\vec{x}_1):}R \\ &= \mathrm{e}^{:-H(\vec{x}_2):}R\end{aligned}$$

We can move the Hamiltonian to another location via the similarity transformation.

We can transform the operator by transforming the generator.

Campbell-Baker-Hausdorff formula

$$\mathrm{e}^{:H_A:}\mathrm{e}^{:H_B:} = \mathrm{e}^{:H:}$$

CBH tells us how to concatenate Hamiltonians

where

$$H = H_A + H_B + \frac{1}{2}[H_A, H_B] + \frac{1}{12}\Big[H_A - H_B, [H_A, H_B]\Big] + \ldots$$

Normal forms

We can propagate a Hamiltonian by propagating its coefficients

$$H^{(1)} = h_i^{(1)} x_i = h_i^{(1)} R_{ij}^{-1} y_j = \tilde{h}^{(1)} y_j$$

$$\tilde{h}^{(1)} = \left(R^{-1}\right)^T h^{(1)} = S^{(1)} h^{(1)}$$

Linear transform:

$$\vec{y} = R\vec{x}$$

To write a map M on its normal form we need to find K and C such that:

$$\mathcal{M} = \mathrm{e}^{:-H:} R = \mathrm{e}^{:-K:} \mathrm{e}^{:-C:} R \mathrm{e}^{:K:}$$

We can re-write as

$$\mathrm{e}^{:-H:} \boxed{R \mathrm{e}^{:-K:} R^{-1}} = \mathrm{e}^{:-K:} \mathrm{e}^{:-C:}$$

A similarity transform! We get:

$$\mathrm{e}^{:-H:} \mathrm{e}^{:-SK:} = \mathrm{e}^{:-K:} \mathrm{e}^{:-C:}$$

This we can write order-by-order:

$$H = H^{(3)} + H^{(4)} + H^{(5)}$$

$$K = K^{(3)} + K^{(4)} + K^{(5)}$$

$$C = C^{(3)} + C^{(4)} + C^{(5)}$$

$$SK = S^{(3)} K^{(3)} + S^{(4)} K^{(4)} + S^{(5)} K^{(5)}$$

Normal forms cont'd

We solve order-by-order $\quad \mathrm{e}^{:-H:}\mathrm{e}^{:-SK:} = \mathrm{e}^{:-K:}\mathrm{e}^{:-C:}$

$$\mathrm{e}^{:-H^{(3)}:}\mathrm{e}^{:-S^{(3)}K^{(3)}:} = \mathrm{e}^{:-K^{(3)}:}\mathrm{e}^{:-C^{(3)}:}$$

$$H = H_A + H_B + \frac{1}{2}[H_A, H_B] + \frac{1}{12}\Big[H_A - H_B, [H_A, H_B]\Big] + \ldots$$

From CBH we get:

$$H^{(3)} + S^{(3)}K^{(3)} = K^{(3)} + C^{(3)} + \text{higher orders}$$

Since $C^{(3)} = 0$ (no tune-shift term of third order) we can write

$$K^{(3)} = (1 - S^{(3)})^{-1}H^{(3)}$$

Keeping all order up to fourth order:

$$H^{(4)} + S^{(4)}K^{(4)} + \frac{1}{2}\Big[H^{(3)}, S^{(3)}K^{(3)}\Big] = K^{(4)} + C^{(4)} + \text{higher orders}$$

We solve for $C^{(4)}$ and $K^{(4)}$:

$$(1 - S^{(4)})K^{(4)} + C^{(4)} = H^{(4)} + \frac{1}{2}\Big[H^{(3)}, S^{(3)}K^{(3)}\Big]$$

In fourth order we have nonzero tune-shift polynomial

Method

$$(1 - S^{(4)})K^{(4)} + C^{(4)} = H^{(4)} + \frac{1}{2}\left[H^{(3)}, S^{(3)}K^{(3)}\right]$$

We cannot invert (1 - $S^{(4)}$) because it has 3 zero eigenvalues. But $S^{(4)}$ is constructed from a pure rotation matrix *R* and these zero eigenvalues corresponds to eigenvector monomials:

$$(x^2 + x'^2)^2 \qquad (y^2 + y'^2)^2 \qquad (x^2 + x'^2)(y^2 + y'^2)$$

which are proportional to:

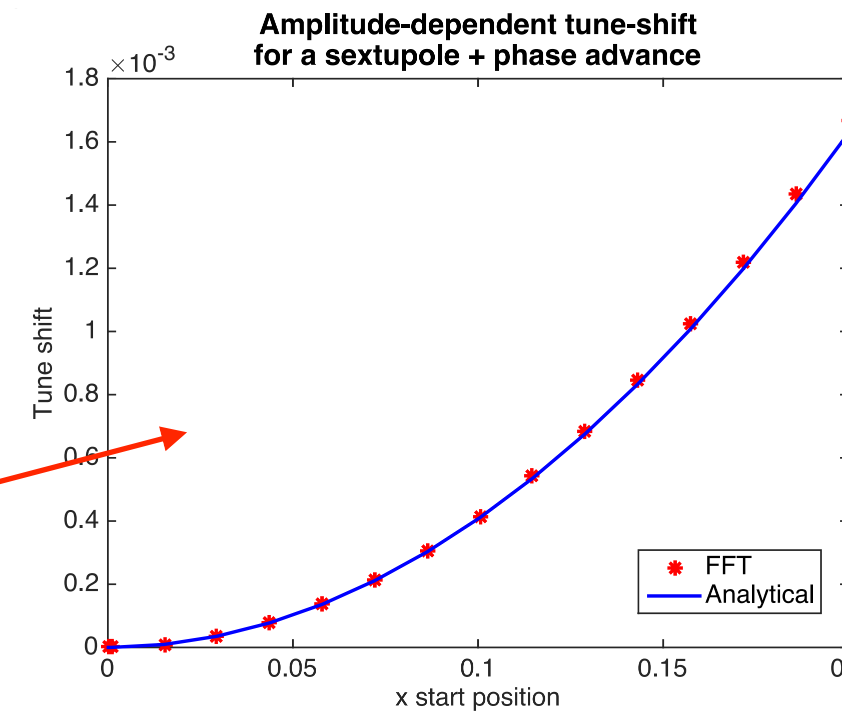
$$J_x^2, \quad J_y^2, \quad J_x J_y$$

We invert (1 - $S^{(4)}$) by SVD and construct a projector corresponding to the zero eigenvalues, i.e. a null sp

$$U\Lambda V^T = (1 - S^{(4)})^{-1} \qquad \mathrm{Pr} = \sum_{\mathrm{eig}=0} \frac{|V><U|}{<V|U>}$$

Then we get $C^{(4)}$ by projecting RHS onto null space:

$$C^{(4)} = \mathrm{Pr}\left\{H^{(4)} + \frac{1}{2}\left[H^{(3)}, S^{(3)}K^{(3)}\right]\right\}$$



Adding octupoles only contributes linearly to fourth order:

$$C^{(4)} = \mathrm{Pr}\left\{\tilde{H}^{(4)} + H^{(4)} + \frac{1}{2}\left[H^{(3)}, S^{(3)}K^{(3)}\right]\right\}$$

To compensate tune-shift: set octuple strengths such RHS = 0.